Math 742 Complex Variables Exam 1

The following problems were featured on qualifying exams in complex analysis at the graduate center. Submit your work on any 5.

1. Show that in an arbitrarily small disk $\{z: |z| < \varepsilon\}$ the function $f(z) = e^{1/z}$ takes every non-zero value infinitely often.

2. Find a conformal mapping from a half open unit disk onto the open unit disk

3. Give an explicit formula for a biholomorphism between the slit unit disk $\mathbb{D} - [0, 1)$ and the half-strip $\{z \in \mathbb{C}: |Im(z)| < 1, Re(z) > 0\}$.

4. Let Ω be a region and let $f, g: \Omega \to \mathbb{C}$ be holomorphic functions satisfying $f(z)g(z) = 0$ for every $z \in \Omega$. Show that either $f \equiv 0$ or $g \equiv 0$.

5. Show that for each $R > 0$, if n is large enough,

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

has no zeros in the disk $\{z: |z| < R\}$.

6. Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

7. Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

8. Show that an entire function f satisfying $|f(z)| \leq 1 + |z|^{1/2}$ for any z must be a constant function.

9. Show that an entire function f with $Re(f) > 0$ must be constant.

10. Suppose $f: D \to \mathbb{C}$ is holomorphic in the unit disk D and the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is 1. Show that f has at least one singular point on the boundary unit circle.

11. Let $f: D \to \mathbb{C}$ be holomorphic in the unit disk D and continuous on the boundary $C = \partial D$. Suppose there is an open arc $I \subset C$ such that $f_I = 0$. Show that $f = 0$ everywhere in D.